Project #3:
Hedging with a Basket of Futures

Raaid Ahmad

Investment Science
Professor Tzitzouris
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A)

At time T, the cash flow from this Basket of Futures is y such that $y = x + (F_t - F_0)h$. In the preceding equation, x is the number of shares of the stock to be hedged times the spot price of the stock at time T. F_t is the futures price for the basket of stocks at time T and F_0 is the futures price for the basket of stocks at time 0. Finally, h is the amount and position of the hedge. In order to prove that the cash flow is $x + (F_t - F_0)h$ the following strategy can be adopted.

At time 0, one must take a long position on d(1,T) futures. d(1,T) represents the discount factor at time 1 for a bond that matures at time T. For every subsequent period of time k, one must increase the long position to d(k+1,T). This increase continues until time T-1 at which point the position is 1.

Time	**Position**
0	d(1,T)
1	d(2,T)
2	d(3,T)
3	d(4,T)
T-1	d(T,T)

At any given time k, the profit garnered from the previous period is $(F_k - F_{k-1}) * d(k,T)$. For the purposes of this proof, the profit generated is re-invested at time k in the interest rate market until time T. The total profit generated by this strategy can be obtained by a summation of the profit from every period from 0 to T-1. The underlined terms in the equation below cancel out, leaving the final result, which represents the total profit generated.

$$(F_T \underline{- F_{T-1}}) + ((\underline{F_{T-1}} \underline{- F_{T-2}})) + \ \dots + ((\underline{F_2} \underline{- F_1})) + ((\underline{F_1} - F_0)) = F_T - F_0$$

The total profit $F_T - F_0$ times h for the total position of the hedge yields $(F_T - F_0)h$ and with the addition of the original payment x, yields the final cash flow y, at time T as $y = x + (F_T - F_0)h$. This proof asserts the real world claim that a basket of futures contracts can be used to hedge a futures contract and eliminate a significant portion of the risk involved in the contract.

B)

As proven in the previous section the cash flow of these transactions is y, such that $y = x + (F_t - F_0)h$. The variance of the cash flow is:

$$\text{var}(y) \Rightarrow E[x - \bar{x} + (F_T - \bar{F}_T)h]^2 \Rightarrow \text{var}(x) + 2\text{cov}(x, F_T)h + \text{var}(F_T)h^2$$

In order to obtain the minimum variance hedge, this function must be minimized. This is accomplished by taking the derivative with respect to h and setting that derivative to zero which yields the equation $0 = 2\text{cov}(x, F_T) + 2\,\text{var}(F_T)h$. Solving for h, the equation below is obtained.

$$h = -\text{cov}(x, F_T)/\text{var}(F_T)$$

The beta of a stock with respect to its industry is denoted by $ß = \text{cov}(S_T, F_T) / \text{var}(F_T)$. As described in part A, $x = \mu S_T$. It follows that $ß_x = \mu\, \text{cov}(x, F_T) / \text{var}(F_T)$. After some algebraic manipulation the final result is found.

$$ß_x = \mu\, \text{cov}(x, F_T) / \text{var}(F_T) \Rightarrow \mu ß_x = \text{cov}(x, F_T) / \text{var}(F_T) \Rightarrow \mu ß_x = -h \Rightarrow$$

$h = -\mu ß_,$ Which represents the minimum variance hedge where ß is the beta of the stock being hedged with respect to the industry.

Though it is sometimes possible to completely eliminate risk with the perfect hedge, often the entirety of the risk cannot be eliminated. This can be for a multitude of reasons including a lack of contracts for the asset to be hedged, non-correlating delivery dates, or even a lack of liquidity in the futures market. The amount of risk that cannot be eliminated due to the aforementioned reasons is called basis risk. Finding the minimum variance hedge allows one to find the hedge that has the lowest basis risk.

C)

The next step in this procedure is to prove that $cov(y,F_I) = 0$. It is given that the beta of the basket of stocks with respect to the industry is 1, yielding the following equations.

$ß = cov(B,I)/var(I) = 1 \quad \Rightarrow \quad var(I) = cov(B,I)$

Using the equation for the variance of y, and canceling like terms, the variance of the cash flow in terms of the variance of the basket and industry is obtained.

$var(y) = var(B) – cov(B,I)^2 / var(I) = var(B) – var(I)$

Finally, using the equation below it can be seen that the variance of the basket is equal to the sum of the variance of the cash flow, the variance of the industry, and 2 times the covariance between the cash flow and the industry.

$var(B) = var(y) + var(I) = var(y) + 2cov(y,I) + var(I)$

This means that 2cov(y,I) must be 0 because the variance of the basket is equal to the variance of the cash flow plus the variance of the industry. This shows that there is no correlation between the cash flow and the futures prices in the industry. This is significant because given no correlation between the cash flow and industry futures prices, the long position discussed earlier has been fully hedged and all industry related risk has been eliminated.

D)

The variance of y, as found in the previous sections, is given below.

$var(y) = var(x) – cov(x, F_T)^2 / var(F_T)$.

Since the variance of the basket, var(B) is equal to the variance of the industry var(I) and the beta of the stock with respect to the basket is equal to the beta of the stock with respect to the industry, the following can be deduced.

$var(y) = var(x) - cov(x,I)^2 / var(I)$
$ß = cov(x,I) / var(I)$

$var(y) = var(x) – [(cov(x,I)^2 / var(I)) * (var(I) / var(I))]$ ← Multiplying by 1

$var(y) = var(x) – [(cov(x,I)^2 * var(I)) / var(I)^2] \rightarrow var(x) – [(cov(x,I)^2 / var(I)^2] * var(I) \rightarrow$
$var(y) = var(x) – var(I) * ß^2 \rightarrow var(x) = var(y) + var(I) * ß^2$

This shows that the variance of y is the diversifiable risk. This step of the process is important because of the purpose of the original basket of stocks. The basket is chosen so that it's behavior mimics the behavior of the entire industry. This fact indicates that in theory, if the basket has been chosen correctly, the variance of the basket and the industry should be equal, or very close. Above, the quantity var(I) * $ß^2$ is shown as one addend in the sum that gives the variance of the future position x. This quantity is known as the systematic risk which is defined as the risk that cannot be diversified away because it is associated with the market itself. The second addend in the sum is the variance of the cash flow y. This variance is known as the specific risk which is directly tied to the risk of the asset itself. The specific risk has 0 correlation with the systematic risk and can be diversified away given the correct hedging technique.

In the case of an unequal basket and industry variance an extra risk term appears.

$var(y) = var(x) - 2cov(x,B) * cov(x,I) / var(I) + var(B) * cov(x,I)^2 / var(I)^2$.

$var(y) = var(x) – \underline{2cov(x,B) * ß} + var(B) * ß^2$. → Underlined term is basis risk.

This extra risk term is the basis risk, discussed in part B. If the basket of futures fails to mimic the market exactly, then the basket cannot perfectly hedge the risk associated with the asset. The difference between the perfect hedge and the hedge produced by the imperfect basket is known as the basis risk.

E)

Assumption 1: The beta of the basket of stocks with respect to the industry is equal to one.

Given this assumption, the covariance of the cash flow y (from part A) and the market-weighted average of the futures prices is 0 because they are uncorrelated. This 0 correlation proves that the risk inherent in the stock has been successfully hedged by the basket of stocks that has been created. The industry-specific risk can be reduced if this assumption turns out to be true. If this assumption is false, then the basket cannot fully hedge the risk of the stock.

Assumption 2 and 3: The variance of the basket is equal to the variance of the industry and The beta of the stock with respect to the basket is equal to ß.

Given these assumptions, it was proven in part D that the variance of the cash flow is what is called the specific risk. The specific risk is the risk inherent to the stock that can be diversified away given the proper hedging technique. The reason for this is that the cash flow variance is uncorrelated with the industry variance.

Assumption 4: The weights in the basket must sum to 1.

This is a fair assumption to make because the purpose of the basket of futures is to mimic the industry performance as a whole. If the sum of the weights in the basket is less than one, the basket mimics only a part of the industry. If this is the case, then the correlation coefficient between the cash flow variance and industry would no longer be 1.